Size distribution and the Hausdorff-Besicovitch dimension of two-scale Cantor dusts

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Simple fractal sets (for example, Cantor dust) can be characterized by a distribution function of sizes of the set's "building blocks." This characterization can be useful in problems of fractal growth and coarsening. We test it on a simple example of a two-scale deterministic Cantor dust. In the limit of $m \gg 1$ (where m is the number of iterations in the fractal generating algorithm), the discrete binomial distribution of sizes of this set can be approximated by a continuous distribution. This continuous distribution gives an accurate estimate for the Hausdorff-Besicovitch dimension. An algorithm is suggested for generating a random two-scale Cantor dust with a tunable fractal dimension. [S1063-651X(99)01901-7]

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There has been a great deal of interest in fractal growth far from equilibrium [1–3]. Inverse processes of fractal coarsening (phase ordering of fractal clusters) have also attracted attention; see Refs. [4–6] and references therein. For most of these systems, a complete theoretical description of the dynamics of fractal clusters is still lacking. A simplistic mean-field approach to these complex problems deals with the evolution of a distribution function of the fractal cluster with respect to the sizes of its elements, or "building blocks" (for example, of drops on a fractal-tree skeleton). Before using this approach for the dynamical problems, however, one has to test its ability to characterize the basic geometrical properties of the system, such as fractal dimension. One of the aims of this paper is to perform this test. We take an example of a deterministic two-scale fractal set (Cantor dust; see Refs. [7,8]) with a tunable (Hausdorff-Besicovitch) fractal dimension. We characterize this set by an approximate continuous size distribution function and show that this distribution gives accurate results for the d measure [7,8] and fractal dimension. (Incidentally, going to the seemingly innocent limit of a Gaussian distribution, one arrives at a wrong estimate for the d measure and we give a brief explanation of this phenomenon.) The second aim of this work is to present a simple algorithm for generating random two-scale Cantor dusts with a tunable fractal dimension. Cantor dust with such properties represents a convenient initial condition for simulating fractal coarsening processes under different transport mechanisms. In particular, since we do not wish to deal with a size distribution that is a δ function, we chose to work with a two-scale Cantor dust.

We employ a well-known algorithm for creating a deterministic two-scale fractal set with a tunable fractal dimension [8]. The initiator of the fractal is a square of a unit side length. The generator consists of n_1 squares of side l_1 and n_2 squares of side l_2 where $l_2 > l_1$. In each step of the fractal's construction every full square is replaced by the (properly rescaled) generator. Thus a two-dimensional two-scale "prefractal" is created. The first two generations of the prefractal are shown in Fig. 1 for a particular choice of the parameters n_1, n_2, l_1, and l_2.

Let the squares of the prefractal be indexed according to their side length in an increasing order. That is, the smallest squares have index $k=0$, whereas the largest ones have index $k=m$, where m is the generation number. The (discrete) distribution of the squares sizes $f_m(k)$ is the binomial distribution

$$f_m(k) = C_m^k \left(\frac{n_1}{n_1+n_2} \right)^{m-k} \left(\frac{n_2}{n_1+n_2} \right)^k . \tag{1}$$

Using this distribution function, one can compute the d measure of the fractal set [7,8]

$$M_d = N_m \sum_{k=0}^{k=m} f_m(k) [l_1^{(m-k)} l_2^k]^d = (n_1 l_1^d + n_2 l_2^d)^m , \tag{2}$$

where $N_m = (n_1+n_2)^m$ is the total number of squares in the mth generation. The Hausdorff-Besicovitch dimension D is defined as the root of the equation $M_d = 1$ [7,8], which yields, for any m,

$$n_1 l_1^D + n_2 l_2^D = 1 . \tag{3}$$

Solving this algebraic equation, one finds D. The four parameters entering Eq. (3) make it possible to tune the fractal dimension on the interval $0 \le D \le 2$.

In the limit of $m \gg 1$ the binomial distribution $f_m(k)$ can be approximated by a continuous distribution function. Let us call it $g_m(R)$ and define it in a usual way

$$f_m(k) \Delta k = g_m(R) dR . \tag{4}$$

Then, using Stirling's formula [9] in Eq. (1) and going over from the index k to the square size R, we obtain

$$g_m(R) = \frac{m^{m+1/2} \kappa^{-\kappa-1/2} (m-\kappa)^{-m+\kappa-1/2}}{(2\pi)^{1/2} \ln(l_2/l_1) R} \times \left(\frac{n_1}{n_1+n_2} \right)^{m-\kappa} \left(\frac{n_2}{n_1+n_2} \right)^{\kappa} , \tag{5}$$

where

$$\kappa = \frac{\ln(R/l_1^m)}{\ln(l_2/l_1)} . \tag{6}$$

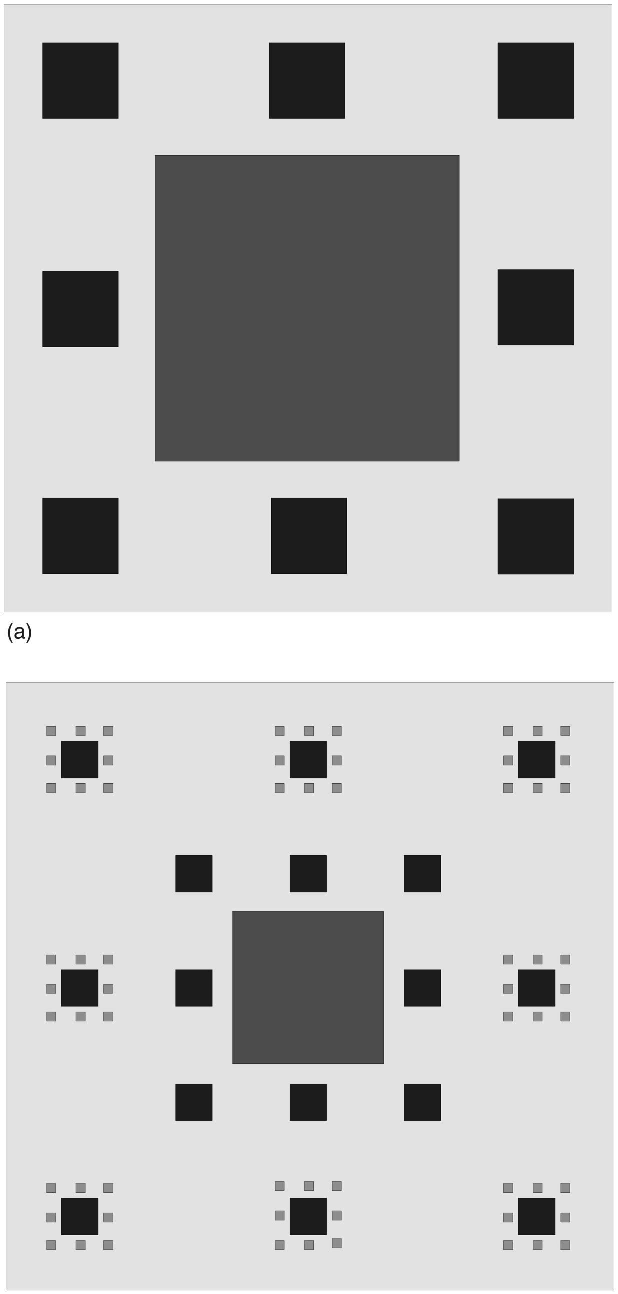

FIG. 1. (a) First and (b) second generations of the two-scale prefractal for $n_1=8$, $n_2=1$, $l_1=1/8$, and $l_2=1/2$.

The d measure of the set can be written now as

$$M_d = N_m \int_{R_{min}(m)}^{R_{max}(m)} g_m(R) R^d \, dR \tag{7}$$

and the Hausdorff-Besicovitch dimension can be found from the equation $M_d=1$. The continuous distribution function gives an accurate value of the d measure. As an illustration we computed M_d from Eqs. (5) and (7) for the same choice of parameters as in Fig. 1 and compared the results with those obtained with the exact binomial distribution (1). The agreement is good, even for moderate m (see Fig. 2). (For these parameters $D=1.260\ldots$).

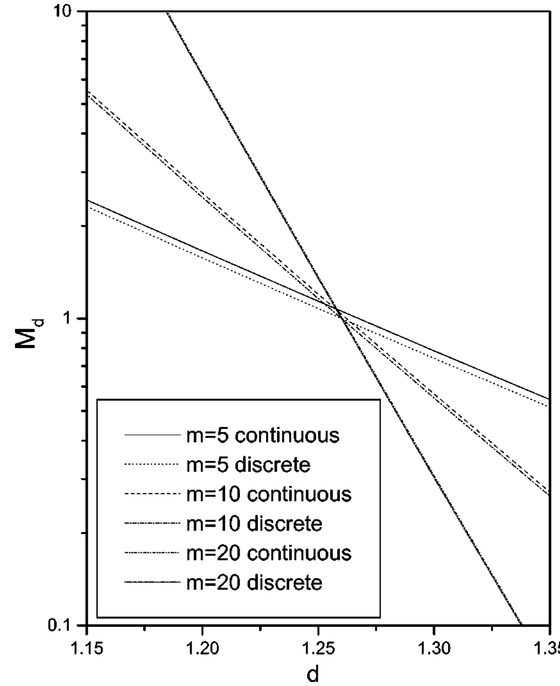

FIG. 2. d measure of the fractal set versus d, computed numerically by using the continuous distribution function (5) for $m=5$, 10, and 20. Also shown are the exact values of the d measure for the same m, calculated by using the binomial distribution function (1). The parameters are the same as in Fig. 1.

A simple saddle-point argument explains this agreement. Indeed, returning to the variable k (but still using the Stirling's formula), one can rewrite M_d as

$$M_d = \int_0^m h_m(k) dk, \tag{8}$$

where

$$h_m(k) = m^{1/2}[2\pi k(m-k)]^{-1/2} \exp[\Phi_m(k)] \tag{9}$$

and

$$\Phi_m(k) = -k \ln(k/m) - (m-k)\ln[(m-k)/m] + (m-k)\ln(n_1 l_1^d) + k \ln(n_2 l_2^d). \tag{10}$$

Using the saddle-point approximation and extending the integration limits to $\pm\infty$, we obtain

$$M_d = \frac{(n_1 l_1^d + n_2 l_2^d)^{m+1}}{(2\pi m n_1 n_2 l_1^d l_2^d)^{1/2}} \int_{-\infty}^{\infty} \exp\left[-\frac{m(k-k_0)^2}{2k_0(m-k_0)}\right] dk, \tag{11}$$

where the saddle point k_0 is given by

$$k_0 = \frac{n_2 l_2^d m}{n_1 l_1^d + n_2 l_2^d}. \tag{12}$$

Calculating the integral, one finally obtains

$$M_d = (n_1 l_1^d + n_2 l_2^d)^m, \tag{13}$$

in agreement with Eq. (2).

A warning should be issued here. Indeed, one is tempted to go a step further and approximate, for large m, the binomial distribution (1) by the Gaussian distribution

$$G_m(k)=\frac{n_1+n_2}{(2\pi m n_1 n_2)^{1/2}} \times \exp\left[-\frac{(n_1+n_2)^2(k-k_*)^2}{2mn_1n_2}\right], \tag{14}$$

where

$$k_*=\frac{n_2 m}{n_1+n_2}. \tag{15}$$

Indeed, it is well known that for $m \gg 1$ the Gaussian distribution (14) is an excellent approximation to the binomial one for calculating moments of k (see, for example, Ref. [10], p. 133). We notice, however, that it is not so for the purpose of computing the d measure. (One can easily check that using the Gaussian distribution one would arrive at a wrong result for the d measure.) The reason is quite simple: The d measure is a convolution, with the binomial distribution, of *exponentials* of large parameters k and $m-k$ [see Eq. (2)] and these exponentials affect significantly the saddle-point calculations.

Our algorithm for generating a *random* two-scale Cantor dust is an extension of a single-scale algorithm by Mandelbrot (see Ref. [7], p. 211) to two-scale fractals. Let $1/l_1$, $1/l_2$, and l_2/l_1 be integers and assume for simplicity that $n_1=n_2=1$. Each iteration in the generating algorithm consists of the following four steps.

(a) Every occupied square is divided into $1/l_2^2$ identical squares with side length $l_2\times$(side length before this iteration).

(b) For each of these $1/l_2^2$ new squares we toss a coin in order to determine its fate. The probability for a square to survive is q.

(c) Each of the squares that were created in step (a) and remained empty after step (b) is divided into l_2^2/l_1^2 identical squares with side length $l_1\times$(side length before this iteration).

(d) For each of the squares generated in step (c) we toss a coin in order to determine its fate. The probability for a square to survive is p.

The probability p_1 that a square with a side length $l_m(k)=l_1^{(m-k)}l_2^k$ survives after m iterations is given by

$$p_1=C_m^k q^k(1-q)^{m-k}p^{m-k}. \tag{16}$$

The probability that $n_m(k)$ squares of side length $l_m(k)$ survive after the mth iteration is given by the binomial distribution

$$P[n_m(k)]=C_{N_m(k)}^{n_m(k)}p_1^{n_m(k)}(1-p_1)^{N_m(k)-n_m(k)}, \tag{17}$$

where $N_m(k)=1/l_m(k)^2$ is the total number of $l_m(k)$ squares (occupied or empty) in the unit square. Therefore, the average number of squares with side length $l_m(k)$ that survive after m iterations is equal to the average of the binomial distribution given by Eq. (17):

$$\overline{n_m(k)}=N_m(k)p_1=C_m^k\left(\frac{q}{l_2^2}\right)^k\left[\frac{(1-q)p}{l_1^2}\right]^{m-k}. \tag{18}$$

The (normalized to unity) size distribution function of the random two-scale Cantor dust is $\overline{n_m(k)}/\overline{N_m}$, where

$$\overline{N_m}=\sum_{k=0}^{m}\overline{n_m(k)} \tag{19}$$

is the average number of surviving squares of all possible side lengths. A natural generalization of the d measure to this fractal set is

$$\overline{M_d}=\sum_{k=0}^{m}\overline{n_m(k)}l_m(k)^d=\sum_{k=0}^{m}C_m^k\left(\frac{q l_2^d}{l_2^2}\right)^k\left[\frac{(1-q)pl_1^d}{l_1^2}\right]^{m-k}. \tag{20}$$

The Hausdorff-Besicovitch dimension is calculated from the equality $\overline{M_d}=1$, which yields

$$(1-q)pl_1^{D-2}+ql_2^{D-2}=1. \tag{21}$$

The four parameters entering Eq. (21) make it possible to tune the fractal dimension on the interval $0\leqslant D\leqslant 2$. In particular, by a proper choice of the survival probabilities q and p, one can generate a random fractal with the same fractal dimension as the deterministic one [compare to Eq. (3) with $n_1=n_2=1$]. A continuous size distribution function can be introduced in the limit of $m\gg 1$, just as in the deterministic case, and it can be used to estimate the Hausdorff-Besicovitch fractal dimension.

In summary, both deterministic and random versions of two-scale Cantor dusts are available for simulating growth and coarsening of multiple-connected fractal objects. The continuous size distribution functions of these sets yield accurate estimates of the fractal dimension. An extension to three dimensions is straightforward. Finally, the square building blocks can be readily replaced in these algorithms by other geometric shapes (for example, by spherical droplets of the same sizes).

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